

# Howson's property for semidirect products of semilattices by groups

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**Abstract.** An inverse semigroup  $S$  is a Howson inverse semigroup if the intersection of finitely generated inverse subsemigroups of  $S$  is finitely generated. Given a locally finite action  $\theta$  of a group  $G$  on a semilattice  $E$ , it is proved that  $E *_\theta G$  is a Howson inverse semigroup if and only if  $G$  is a Howson group. It is also shown that this equivalence fails for arbitrary actions.

**Keywords.**  $E$ -unitary inverse semigroup, Howson's theorem, locally finite action, semidirect product of a semilattice by a group.

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## 1. INTRODUCTION

In [7], Howson proved a result that would become known as Howson's Theorem:

**Howson's Theorem.** *The intersection of two finitely generated subgroups of a free group is a finitely generated subgroup.*

This property, not being true in general, led to defining a group  $G$  to be a *Howson group* if the intersection of any two finitely generated subgroups of  $G$  is again a finitely generated subgroup of  $G$ .

Similarly, we say that an inverse semigroup  $S$  is a *Howson inverse semigroup* if the intersection of any two finitely generated inverse subsemigroups of  $S$  is finitely generated. Note that if  $S$  is a group, then the inverse subsemigroups of  $S$  are precisely its subgroups, thus  $S$  is a Howson inverse semigroup if and only if it is a Howson group.

We remark also that  $S$  being a Howson inverse semigroup does not imply that the intersection of finitely generated *subsemigroups* of  $S$  is finitely generated (i.e.,  $S$  needs not be a *Howson semigroup*). A counterexample is provided by the free group  $F$  of rank 2 [5, Proposition 2.1(ii)].

Contrary to the behaviour of free groups, Jones and Trotter showed that, although the free monogenic inverse semigroup is a Howson inverse semigroup [11, Theorem 1.6], that is not the case for any other free inverse semigroup [11, Corollary 2.2]. However, the intersection of any two monogenic inverse subsemigroups of a free inverse semigroup is always finitely generated [13] and the intersection of any two closed inverse submonoids of the free inverse monoid, both finitely generated as closed inverse monoids, is finitely generated [8].

As one would expect, no general characterizations of Howson groups are known. In what is probably the most general result of that kind, Araújo, Sykiotis and the first author proved that every fundamental group of a finite graph of groups with virtually polycyclic vertex groups and finite edge groups is a Howson group [2, Theorem 3.10].

Thus the problem of identifying Howson inverse semigroups promises to be even harder. Given the extraordinary importance assumed by  $E$ -unitary inverse semigroups in the theory of inverse semigroups, they constitute a good starting point, particularly the case of semidirect products of semilattices by groups.

Indeed, O'Carroll proved in [12] that every  $E$ -unitary inverse semigroup  $S$  embeds into some semidirect product  $E *_\theta G$  of a semilattice by a group. The second author proved in [14] that this embedding can be assumed to be normal-convex, i.e. every quotient of  $S$  embeds in some quotient of  $E *_\theta G$ . Therefore it is a natural problem to determine under which conditions a semidirect product of a semilattice by a group is a Howson inverse semigroup. If the action of  $G$  on  $E$  has a fixed point (i.e. if  $G \cdot e = \{e\}$  for some  $e \in E$ ) then  $G$  embeds in  $E *_\theta G$  and so  $G$  being a Howson group is a necessary condition (cf. Lemma 3.3). We note that if  $E$  has an identity (maximum) or a zero (minimum) then such an element is necessarily a fixed point for any action of a group.

In Section 3, we show that if  $E$  is a finite semilattice, then  $E *_\theta G$  is a Howson inverse semigroup if and only if  $G$  is a Howson group. We also prove a theorem on polynomial bounds, introducing the concept of a polynomially Howson inverse semigroup. The main theorem of Section 3 is extended in Section 4 to arbitrary semilattices, provided that the group action is locally finite. Finally, in Section 5, examples are produced to show that anything can happen when the action is not locally finite.

## 2. PRELIMINARIES

Let  $E$  be a  $(\wedge)$ -semilattice and  $G$  a group acting on the left on  $E$  via the homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . As usual, we write  $\theta_g$  instead of  $\theta(g)$  and  $g \cdot e$  instead of  $\theta_g(e)$ , for any  $g \in G$  and  $e \in E$ . In particular,  $\theta$  being a homomorphism is equivalent to  $\theta_{gh}(e) = \theta_g(\theta_h(e))$ , for any  $g, h \in G$  and  $e \in E$ , that is, to  $(gh) \cdot e = g \cdot (h \cdot e)$ .

The action  $\theta$  determines the *semidirect product*  $E *_\theta G$ , where

$$(e, g)(f, h) = (e \wedge (g \cdot f), gh),$$

for all  $(e, g), (f, h) \in E \times G$ . Also recall that  $(e, g)^{-1} = (g^{-1} \cdot e, g^{-1})$ , for each  $(e, g)$ . If the action is trivial, i.e.  $\theta_g = \text{id}_E$  for every  $g \in G$ , then we have the direct product  $E \times G$ ; if the action is obvious, we write simply  $E * G$ .

Let  $\sigma: E *_\theta G \rightarrow E$  and  $\gamma: E *_\theta G \rightarrow G$  denote the projections; thus,  $u = (\sigma(u), \gamma(u))$  whenever  $u \in E *_\theta G$ . Note that, except when  $\theta$  is trivial, only  $\gamma$  is a homomorphism.

Given an inverse semigroup  $S$  and a subset  $X \subseteq S$ , we denote by  $\langle X \rangle$  the inverse subsemigroup of  $S$  generated by  $X$ . In particular, if  $S$  is a group (respectively semilattice),  $\langle X \rangle$  is the subgroup (respectively subsemilattice) of  $S$  generated by  $X$ . For a finitely generated inverse semigroup  $S$ , the *rank* of  $S$  is defined as

$$\text{rk}(S) = \min\{|X| : S = \langle X \rangle\}.$$

If  $A$  is a finite nonempty alphabet, a finite  $A$ -*automaton* is a quadruple of the form  $\mathcal{A} = (Q, q_0, T, \Gamma)$ , where  $Q$  is a finite set,  $q_0 \in Q$ ,  $T \subseteq Q$  and  $\Gamma \subseteq Q \times A \times Q$ . As usual, the elements of  $Q$  will be called *vertices*. A *path* in  $\mathcal{A}$  is a sequence of the form

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} p_n, \quad (1)$$

with  $n \geq 1$  and  $(p_{i-1}, a_i, p_i) \in \Gamma$  for  $i = 1, \dots, n$ . Note that we are not admitting empty paths in this paper. The path (1) has *length*  $n \geq 1$  and *label*  $a_1 a_2 \dots a_n \in A^+$ . It is *successful* if  $p_0 = q_0$  and  $p_n \in T$ . The *language* of  $\mathcal{A}$ , denoted by  $L(\mathcal{A})$ , is the subset of  $A^+$  consisting of the labels of all successful paths in  $\mathcal{A}$ . A language  $L \subseteq A^+$  is *rational* if  $L = L(\mathcal{A})$  for some finite  $A$ -automaton  $\mathcal{A}$ .

A finite  $A$ -automaton  $\mathcal{A} = (Q, q_0, T, \Gamma)$  is:

- *deterministic* if  $(p, a, q), (p, a, q') \in \Gamma$  implies  $q = q'$  for all  $p, q, q' \in Q$  and  $a \in A$ ;
- *complete* if for all  $p \in Q$  and  $a \in A$  there exists some  $q \in Q$  such that  $(p, a, q) \in \Gamma$ .

Let  $S$  be a semigroup. We say that  $X \subseteq S$  is a *rational subset* of  $S$  if there exists a finite alphabet  $A$ , a homomorphism  $\varphi: A^+ \rightarrow S$  and a rational language  $L \subseteq A^+$  such that  $X = \varphi(L)$ . The following result, proved by Anisimov and Seifert in [1], will be important for us:

**Theorem 2.1.** *Let  $G$  be a group and let  $H$  be a subgroup of  $G$ . Then  $H$  is a rational subset of  $G$  if and only if  $H$  is finitely generated.*

For more details on languages, automata and rational subsets, the reader is referred to [4].

### 3. FINITE SEMILATTICES

In this section we consider a group  $G$  acting on the left on a finite semilattice  $E$ . We start with a useful lemma, proved with some help from automata theory.

**Lemma 3.1.** *Let  $E$  be a finite semilattice and  $G$  a group acting on the left on  $E$  via the homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . Let  $X$  be a finite nonempty subset of  $E *_\theta G$  and let  $e \in E$ . Let*

$$S(e) = \{u \in \langle X \rangle : \sigma(u) \geq e, \theta_{\gamma(u)} = id_E\}.$$

*Then if  $S(e)$  is nonempty, its projection  $\gamma(S(e))$  is a finitely generated subgroup of  $G$ .*

*Proof.* For simplicity, write  $S$  instead of  $S(e)$ . Assume that  $S \neq \emptyset$ . It is straightforward to show that  $S$  is an inverse subsemigroup of  $E *_\theta G$ , so the proof is omitted.

We show that  $S$  is a rational subset of  $E *_\theta G$ . We may assume that  $X = X^{-1}$ , and we introduce a finite alphabet  $A = \{a_x : x \in X\}$ . Let  $\varphi: A^+ \rightarrow E *_\theta G$  be the homomorphism defined by  $\varphi(a_x) = x$ . Let  $\mathcal{A}$  be the Cayley automaton for  $E *_\theta \text{Aut}(E)$  determined by  $A$ , that is, the  $A$ -automaton  $\mathcal{A} = (Q, q_0, T, \Gamma)$  with

$$\begin{aligned} Q &= \{q_0\} \dot{\cup} (E \times \text{Aut}(E)), \\ T &= \{f \in E : f \geq e\} \times \{id_E\}, \\ \Gamma &= \{(q_0, a_x, (\sigma(x), \theta_{\gamma(x)})) : x \in X\} \\ &\quad \cup \{((f, \pi), a_x, (f \wedge \pi(\sigma(x)), \pi\theta_{\gamma(x)})) : f \in E, \pi \in \text{Aut}(E), x \in X\}. \end{aligned}$$

It follows that  $\mathcal{A}$  is complete and deterministic and, in particular, that for every  $v \in A^+$ , there exists a unique  $q_v \in Q$  such that  $q_0 \xrightarrow{v} q_v$  is a path in  $\mathcal{A}$ , namely

$$q_v = (\sigma(\varphi(v)), \theta_{\gamma(\varphi(v))}). \quad (2)$$

But then, for every  $v \in A^+$ , we have that  $v$  labels a successful path in  $\mathcal{A}$  if and only if  $\sigma(\varphi(v)) \geq e$  and  $\theta_{\gamma(\varphi(v))} = id_E$ , that is, if and only if  $\varphi(v) \in S$ . Therefore  $S = \varphi(L(\mathcal{A}))$ , and so  $S$  is a rational inverse subsemigroup of  $E *_\theta G$ .

Composing  $\varphi$  with  $\gamma$ , we deduce that  $\gamma(S)$  is a rational inverse subsemigroup of  $G$ , i.e. a rational subgroup of  $G$ . By Theorem 2.1,  $\gamma(S)$  is finitely generated.  $\square$

Before we proceed, we make the following observation.

**Remark 3.2.** The actual proof of Anisimov and Seifert's Theorem (or rather, a few proofs among many) allows us to draw some conclusions on the rank of  $\gamma(S(e))$ .

Indeed, if a subgroup  $H$  of a group  $G$  is rational, we may write  $H = \rho(L(\mathcal{B}))$  for some (finite)  $(B \cup B^{-1})$ -automaton  $\mathcal{B}$  and homomorphism  $\rho: (B \cup B^{-1})^* \rightarrow G$  satisfying  $\rho(b^{-1}) = (\rho(b))^{-1}$  for every  $b \in B$ . Then  $H$  is generated by  $\rho(Y)$  where  $Y$  consists of all words in  $\rho^{-1}(H)$  of length less than twice the number of vertices of  $\mathcal{B}$  (cf. [3, Theorem 3.1]). As  $|Y| \leq \sum_{i=0}^{2m-1} (2|B|)^i$ , where  $m$  denotes the number of vertices of  $\mathcal{B}$ , we have that

$$\text{rk}(H) \leq |\rho(Y)| \leq \sum_{i=0}^{2m-1} (2|B|)^i = \frac{1 - (2|B|)^{2m}}{1 - 2|B|}.$$

Assume that  $S(e) \neq \emptyset$ . As  $\gamma(S(e)) = (\gamma\varphi)(L(\mathcal{A}))$ , the automaton  $\mathcal{A}$  has

$$|E| \times |\text{Aut}(E)| \leq |E| \times (|E| - 1)! = |E|!$$

vertices, and its alphabet of labels has size  $|X|$ , we deduce that

$$\text{rk}(\gamma(S(e))) \leq \sum_{i=0}^{2|E|!-1} (2|X|)^i = \frac{1 - (2|X|)^{2 \times |E|!}}{1 - 2|X|}. \quad (3)$$

We continue with another lemma.

**Lemma 3.3.** *Let  $G$  be a group acting on the left on a semilattice  $E$  via the homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . If  $\theta$  has a fixed point and  $E *_\theta G$  is a Howson inverse semigroup, then  $G$  is a Howson group.*

*Proof.* Let  $e \in E$  be such that  $g \cdot e = e$  for every  $g \in G$ . Then  $G$  is isomorphic to the inverse subsemigroup  $\{e\} \times G$  of  $E *_\theta G$ . Since  $E *_\theta G$  is a Howson inverse semigroup, so is  $G$ . Since the inverse subsemigroups of a group are its subgroups,  $G$  is a Howson group.  $\square$

Now we can prove the main result of this section.

**Theorem 3.4.** *Let  $E$  be a finite semilattice and  $G$  a group acting on the left on  $E$  via the homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . Then the following conditions are equivalent:*

- (i)  $E *_\theta G$  is a Howson inverse semigroup;
- (ii)  $G$  is a Howson group.

*Proof.* (i)  $\Rightarrow$  (ii). Since  $E$  is finite, it has a zero  $0$ , which is necessarily a fixed point of  $\theta$ . Thus we may apply Lemma 3.3.

(ii)  $\Rightarrow$  (i). Let  $X_1, X_2 \subseteq E *_\theta G$  be finite. We build a finite generating set  $X$  for  $\langle X_1 \rangle \cap \langle X_2 \rangle$  as follows.

For  $i = 1, 2$ , let

$$P_i = \{(\sigma(u), \theta_{\gamma(u)}) : u \in \langle X_i \rangle\} \subseteq E \times \text{Aut}(E).$$

For all  $(e, \pi) \in P_1 \cap P_2$  and  $i = 1, 2$ , let

$$\begin{aligned} S_i(e, \pi) &= \{u \in \langle X_i \rangle : \sigma(u) \geq \pi^{-1}(e), \theta_{\gamma(u)} = id_E\}, \\ S'_i(e, \pi) &= \{u \in \langle X_i \rangle : \sigma(u) = e, \theta_{\gamma(u)} = \pi\}, \\ S'(e, \pi) &= S'_1(e, \pi) \cap S'_2(e, \pi), \\ H(e, \pi) &= \gamma(S_1(e, \pi)) \cap \gamma(S_2(e, \pi)). \end{aligned}$$

Write also

$$\begin{aligned} P &= \{(e, \pi) \in P_1 \cap P_2 : S'(e, \pi) \neq \emptyset\}, \\ Q &= \{(e, \pi) \in P_1 \cap P_2 : H(e, \pi) \neq \emptyset\}. \end{aligned}$$

By Lemma 3.1,  $\gamma(S_i(e, \pi))$  is a finitely generated subgroup of  $G$  for all  $(e, \pi) \in Q$  and  $i = 1, 2$ .

Now, since  $G$  is a Howson group, then  $H(e, \pi)$  is also a finitely generated subgroup of  $G$  for every  $(e, \pi) \in Q$ . Let  $Y(e, \pi)$  denote a finite generating set (closed under inversion) of  $H(e, \pi)$ , and let  $X(e, \pi) = \{\pi^{-1}(e)\} \times Y(e, \pi)$ .

Finally, for every  $(e, \pi) \in P$ , we fix some element  $w_{(e, \pi)} \in S'(e, \pi)$ . We claim that the finite set

$$X = \left( \bigcup_{(e, \pi) \in Q} X(e, \pi) \right) \cup \{w_{(e, \pi)} : (e, \pi) \in P\} \quad (4)$$

generates  $\langle X_1 \rangle \cap \langle X_2 \rangle$ .

Let  $(e, \pi) \in Q$  and  $(\pi^{-1}(e), g) \in X(e, \pi)$ . Let  $i \in \{1, 2\}$ . Since  $(e, \pi) \in Q \subseteq P_i$ , we have  $(e, \pi) = (\sigma(u_i), \theta_{\gamma(u_i)})$  for some  $u_i \in \langle X_i \rangle$ . Writing  $h_i = \gamma(u_i)$ , we have  $(e, h_i) \in \langle X_i \rangle$ , hence  $(h_i^{-1} \cdot e, h_i^{-1}) = (e, h_i)^{-1} \in \langle X_i \rangle$ . Since

$$\pi^{-1}(e) = \theta_{\gamma(u_i)}^{-1}(e) = \theta_{h_i}^{-1}(e) = \theta_{h_i^{-1}}(e) = h_i^{-1} \cdot e,$$

we get

$$(\pi^{-1}(e), 1) = (h_i^{-1} \cdot e, 1) = (h_i^{-1} \cdot e, h_i^{-1})(e, h_i) \in \langle X_i \rangle. \quad (5)$$

Now  $(\pi^{-1}(e), g) \in X(e, \pi)$  yields  $g \in \gamma(S_i(e, \pi))$ , hence there exists some  $v_i \in S_i(e, \pi) \subseteq \langle X_i \rangle$  such that  $g = \gamma(v_i)$ . Moreover,  $\sigma(v_i) \geq \pi^{-1}(e)$ . Writing  $f = \sigma(v_i)$ , it follows that  $(f, g) = (\sigma(v_i), \gamma(v_i)) = v_i \in \langle X_i \rangle$ . Since  $f = \sigma(v_i) \geq \pi^{-1}(e)$ , and in view of (5), we get

$$(\pi^{-1}(e), g) = (\pi^{-1}(e), 1)(f, g) \in \langle X_i \rangle.$$

Since  $i \in \{1, 2\}$  is arbitrary, it follows that  $X(e, \pi) \subseteq \langle X_1 \rangle \cap \langle X_2 \rangle$  for every  $(e, \pi) \in Q$ .

On the other hand, it follows from the definitions that  $S'(e, \pi) \subseteq \langle X_1 \rangle \cap \langle X_2 \rangle$ , hence  $w_{(e, \pi)} \in \langle X_1 \rangle \cap \langle X_2 \rangle$  for every  $(e, \pi) \in P$ . Therefore  $X \subseteq \langle X_1 \rangle \cap \langle X_2 \rangle$ .

Conversely, suppose that  $u \in \langle X_1 \rangle \cap \langle X_2 \rangle$ . Write  $e = \sigma(u)$ ,  $g = \gamma(u)$  and  $\pi = \theta_g$ . Then  $u \in S'(e, \pi)$ , hence  $(e, \pi) \in P$  and so  $w = w_{(e, \pi)} \in X$ . Thus we may write  $w = (e, h)$  for some  $h \in G$  satisfying  $\theta_h = \pi = \theta_g$ . Therefore  $\theta_{h^{-1}g} = id_E$ .

On the other hand,

$$(\pi^{-1}(e), h^{-1}g) = (\pi^{-1}(e), h^{-1})(e, g) = (e, h)^{-1}(e, g) = w^{-1}u \in \langle X_1 \rangle \cap \langle X_2 \rangle ,$$

yields

$$(\pi^{-1}(e), h^{-1}g) \in S_1(e, \pi) \cap S_2(e, \pi) .$$

Thus  $h^{-1}g \in H(e, \pi)$  and so we may write  $h^{-1}g = y_1 \dots y_n$  for some  $y_j \in Y(e, \pi)$ , yielding  $(\pi^{-1}(e), y_j) \in X(e, \pi)$  for  $j = 1, \dots, n$ . We show that

$$(\pi^{-1}(e), h^{-1}g) = (\pi^{-1}(e), y_1) \dots (\pi^{-1}(e), y_n) . \quad (6)$$

Indeed,  $y_j \in Y(e, \pi) \subseteq \gamma(S_1(e, \pi)) \cap \gamma(S_2(e, \pi))$  implies that  $\theta_{y_j} = id_E$  for every  $j$ , hence (6) follows from the decomposition  $h^{-1}g = y_1 \dots y_n$ .

Therefore

$$u = (e, g) = (e, h)(h^{-1} \cdot e, h^{-1}g) = w(\pi^{-1}(e), h^{-1}g) \in \langle X \rangle$$

and so  $\langle X \rangle = \langle X_1 \rangle \cap \langle X_2 \rangle$  as claimed. Therefore  $\langle X_1 \rangle \cap \langle X_2 \rangle$  is finitely generated and so  $E *_\theta G$  is a Howson inverse semigroup.  $\square$

In his 1954 paper on the intersection of finitely generated free groups [7], Howson also provided an upper bound on the rank of  $H_1 \cap H_2$  in terms of the ranks of the (nontrivial) subgroups  $H_1$  and  $H_2$ , namely:

$$\text{rk}(H_1 \cap H_2) \leq 2 \text{rk}(H_1) \text{rk}(H_2) - \text{rk}(H_1) - \text{rk}(H_2) + 1 ,$$

which we will refer to as the ‘‘Howson inequality’’. Two years later, Hanna Neumann [10] improved this upper bound to

$$\text{rk}(H_1 \cap H_2) \leq 2(\text{rk}(H_1) - 1)(\text{rk}(H_2) - 1) + 1$$

and conjectured that the factor 2 could in fact be removed from the inequality, in what would become known as the ‘‘Hanna Neumann Conjecture’’. In its full generality, the conjecture would only be proved in 2012, independently by Friedman [6] and Mineyev [9].

We say that an inverse semigroup  $S$  is *polynomially Howson* if there exists a polynomial  $p(x) \in \mathbb{R}[x]$  such that

$$\text{rk}(T_1), \text{rk}(T_2) \leq n \Rightarrow \text{rk}(T_1 \cap T_2) \leq p(n)$$

for all inverse subsemigroups  $T_1, T_2$  of  $S$  and  $n \in \mathbb{N}$ . If  $p(x)$  can be taken to be quadratic, we say that  $S$  is *quadratically Howson*. Note that by the Howson inequality free groups are quadratically Howson.

**Theorem 3.5.** *Let  $E$  be a finite semilattice and  $G$  a polynomially Howson group acting on the left on  $E$  via the homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . Then  $E *_\theta G$  is polynomially Howson.*

*Proof.* Let  $p(x) \in \mathbb{R}[x]$  be such that

$$\text{rk}(H_1), \text{rk}(H_2) \leq n \Rightarrow \text{rk}(H_1 \cap H_2) \leq p(n)$$

for all subgroups  $H_1, H_2$  of  $G$  and  $n \in \mathbb{N}$ .

Let  $X_1, X_2 \subseteq E *_\theta G$  have at most  $n$  elements. Write  $T_i = \langle X_i \rangle$  for  $i = 1, 2$ . By the proof of Theorem 3.4, their intersection  $T_1 \cap T_2$  is generated by  $X = Y \cup W$ , where

$$Y = \bigcup_{(e, \pi) \in Q} X(e, \pi) \quad \text{and} \quad W = \{w_{(e, \pi)} : (e, \pi) \in P\}$$

(cf. (4)). Since  $P, Q \subseteq E \times \text{Aut}(E)$ , we have  $|P|, |Q| \leq |E|!$ . Thus, in particular,  $|W| \leq |E|!$ . Moreover, for each  $(e, \pi) \in Q$ , we have  $|X(e, \pi)| = |Y(e, \pi)| = \text{rk}(\gamma(S_1(e, \pi)) \cap \gamma(S_2(e, \pi)))$ , where, by Remark 3.2,

$$\text{rk}(\gamma(S_i(e, \pi))) \leq q(2\text{rk}(T_i)) \leq q(2n)$$

for  $q(x) = \sum_{j=0}^{2 \times |E|! - 1} x^j$ .

Thus,

$$\text{rk}(\gamma(S_1(e, \pi)) \cap \gamma(S_2(e, \pi))) \leq p(q(2n))$$

and so

$$\text{rk}(T_1 \cap T_2) \leq |E|!(1 + p(q(2n))).$$

Therefore  $S$  is polynomially Howson.  $\square$

**Corollary 3.6.** *Let  $E$  be a finite semilattice and  $G$  a group acting on the left on  $E$  via the homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . If  $G$  is the fundamental group of a finite graph of groups with virtually polycyclic vertex groups and finite edge groups, then  $E *_\theta G$  is polynomially Howson.*

*Proof.* By [2, Theorem 3.10],  $G$  is quadratically Howson, hence we may apply Theorem 3.5.  $\square$

#### 4. LOCALLY FINITE ACTIONS

Let  $E$  be a semilattice and  $G$  a group acting on  $E$ . Given a subgroup  $H \leq G$  and  $e \in E$ , we say that  $H \cdot e = \{h \cdot e : h \in H\}$  is the  $H$ -orbit of  $e$ . The action of  $G$  on  $E$  is said to be *locally finite* if all the  $H$ -orbits are finite whenever  $H$  is a finitely generated subgroup of  $G$ .

We can extend Theorem 3.4 to the following result.

**Theorem 4.1.** *Let  $E$  be a semilattice and  $G$  a group acting on the left on  $E$  by means of a locally finite action  $\theta: G \rightarrow \text{Aut}(E)$ . Then the following conditions are equivalent:*

- (i)  $E *_\theta G$  is a Howson inverse semigroup;
- (ii)  $G$  is a Howson group.



*Proof.* (i)  $\Rightarrow$  (ii). Let  $H, H'$  be finitely generated subgroups of  $G$  and let  $G' = \langle H \cup H' \rangle \leq G$ . We fix some  $e \in E$ . Since  $\theta$  is locally finite,  $G' \cdot e$  is a finite subset of  $E$ . Let  $E'$  denote the (finite) subsemilattice of  $E$  generated by  $G' \cdot e$ . We claim that  $G' \cdot E' \subseteq E'$ .

Indeed, the elements of  $E'$  are of the form  $(g'_1 \cdot e) \wedge \dots \wedge (g'_n \cdot e)$ , with  $g'_i \in G'$ . If  $g' \in G'$ , then

$$g' \cdot ((g'_1 \cdot e) \wedge \dots \wedge (g'_n \cdot e)) = (((g'_1 g'_1) \cdot e) \wedge \dots \wedge ((g'_n g'_n) \cdot e)) \in E'$$

and so  $G' \cdot E' \subseteq E'$ .

Let  $\theta': G' \rightarrow \text{Aut}(E')$  be defined by  $\theta'_{g'} = \theta_{g'}|_{E'}$  for  $g' \in G'$ . Since  $G' \cdot E' \subseteq E'$  and  $(\theta'_{g'})^{-1} = \theta'_{(g')^{-1}}$ , it is easy to check that  $\theta'$  is a well-defined group homomorphism. Moreover, there is a natural embedding of  $E' *_{\theta'} G'$  into  $E *_{\theta} G$ . Thus  $E' *_{\theta'} G'$  is a Howson inverse semigroup.

Since  $E'$  is finite, it has a zero  $0$ , which is necessarily a fixed point of  $\theta'$ . By Lemma 3.3,  $G'$  is a Howson group. Since  $H, H' \subseteq G'$ , it follows that  $H \cap H'$  is finitely generated. Therefore  $G$  is a Howson group.

(ii)  $\Rightarrow$  (i). Let  $X_1, X_2$  be two finite nonempty subsets of  $E *_{\theta} G$ . We may assume that  $X_i = X_i^{-1}$  for  $i = 1, 2$ . Let

$$Y = \{\gamma(x) : x \in X_1 \cup X_2\}, \quad F = \{\sigma(x) : x \in X_1 \cup X_2\}.$$

Let  $H$  be the subgroup of  $G$  generated by  $Y$ . Since  $G$  is a Howson group, so is  $H$ . Let  $E'$  be the subsemilattice of  $E$  generated by  $H \cdot F = \{h \cdot f : h \in H, f \in F\}$ . Since  $\theta$  is locally finite,  $H \cdot F$  is a finite subset of  $E$ . Since finitely generated semilattices are finite, it follows that  $E'$  is finite. An argument as the one above shows that  $H \cdot E' \subseteq E'$ .

Again as in the proof of the direct implication,  $\theta$  induces an action  $\theta': H \rightarrow \text{Aut}(E')$  and we may view  $E' *_{\theta'} H$  as an inverse subsemigroup of  $E *_{\theta} G$ .

Now we note that  $X_i \subseteq F \times Y \subseteq E' *_{\theta'} H$  implies  $\langle X_i \rangle \subseteq E' *_{\theta'} H$  for  $i = 1, 2$ . Since  $E'$  is finite and  $H$  is Howson, we may use Theorem 3.4 to deduce that  $\langle X_1 \rangle \cap \langle X_2 \rangle$  is finitely generated. Therefore  $E *_{\theta} G$  is a Howson inverse semigroup.  $\square$

We discuss now some examples.

**Example 4.2.** A trivial example of a locally finite action is that of a trivial action, that is, an action in which  $g \cdot e = e$ , for all  $e \in E$  and  $g \in G$ . Therefore the direct product of a semilattice by a Howson group is always a Howson inverse semigroup.

If  $G$  is a *locally finite* group (i.e. every finitely generated subgroup of  $G$  is finite), then  $G$  is trivially a Howson group and the action of  $G$  on any semilattice is obviously locally finite. Thus we obtain the following consequence.

**Corollary 4.3.** *Let  $G$  be a locally finite group acting on the left on a semilattice  $E$  via a homomorphism  $\theta: G \rightarrow \text{Aut}(E)$ . Then  $E *_\theta G$  is a Howson inverse semigroup.*

The following example illustrates such a case.

**Example 4.4.** Let  $FS_A$  be the free semilattice on a nonempty set  $A$  and let  $S_A^f$  be the set of all permutations on  $A$  with finite support. Then  $S_A^f$  is a locally finite group and so the natural action of  $S_A^f$  on  $FS_A$  is locally finite.

Indeed,  $FS_A$  is the set of all finite nonempty subsets of  $A$ , endowed with the union operation, and  $S_A^f$  consists of all the permutations of  $A$  which fix all but finitely many elements of  $A$ . Then  $S_A^f$  acts on  $FS_A$  by restriction, and is obviously locally finite.

Let  $E$  be a semilattice with identity 1. We say that  $E$  is *finite above* if  $\{f \in E: f \geq e\}$  is finite for every  $e \in E$ . Given such a semilattice  $E$ , we define its *height function* as the function  $\lambda: E \rightarrow \mathbb{N}$  defined by

$$\lambda(e) = \max\{n \in \mathbb{N}: \text{there exists a chain } 1 = e_0 > \dots > e_n = e \text{ in } E\}.$$

We say that  $E$  is *strongly finite above* if it is finite above and  $\lambda^{-1}(n)$  is finite for every  $n \in \mathbb{N}$ .

**Lemma 4.5.** *The action of any group on a strongly finite above semilattice with identity is locally finite.*

*Proof.* Indeed, it is straightforward to check that  $\lambda(g \cdot e) = \lambda(e)$  for all  $g \in G$  and  $e \in E$ . This can be achieved by induction on  $\lambda^{-1}(n)$ , starting with  $\lambda^{-1}(0) = \{1\}$ , where 1 denotes the identity of  $E$ . Therefore  $G \cdot e \subseteq \lambda^{-1}(\lambda(e))$  and the action is locally finite.  $\square$

We provide next an example of a nontrivial action where:

- the semilattice is infinite, has an identity and is strongly finite above;
- the group is Howson but not locally finite.

**Example 4.6.** Let

$$E = \{(2n+1, 0): n \geq 0\} \cup \left( \bigcup_{n \geq 1} \{2n\} \times \mathbb{Z}/n\mathbb{Z} \right),$$

partially ordered by

$$(k, x) \leq (\ell, y) \text{ if } k < \ell \text{ or } (k, x) = (\ell, y).$$

Then  $E$  is clearly infinite with identity  $(0, 1)$  and finite above, and the height function is given by  $\lambda(k, x) = k - 1$ . Thus  $E$  is strongly finite above. Let  $G$  be the additive group  $\mathbb{Z}$ , which is Howson but not locally finite. We define a (nontrivial) action  $\theta: G \rightarrow \text{Aut}(E)$  by

$$\theta_m(2n+1, 0) = (2n+1, 0), \quad \theta_m(2n, k + n\mathbb{Z}) = (2n, m + k + n\mathbb{Z}).$$

It is straightforward to check that  $\theta$  is well defined, hence it is locally finite by Lemma 4.5. Therefore  $E *_\theta G$  is a Howson inverse semigroup by Theorem 4.1.

## 5. NON LOCALLY FINITE ACTIONS

The next two examples show that for non locally finite actions  $E *_\theta G$  may be a Howson inverse semigroup or not.

**Example 5.1.** By O’Carroll’s construction [12], any free inverse semigroup  $F$  embeds in a semidirect product  $S$  of a semilattice by a free group. If  $F$  is not monogenic, then  $F$  is not a Howson inverse semigroup by [11, Corollary 2.2], therefore  $S$  is not a Howson inverse semigroup either. However, free groups are Howson groups. In view of Theorem 4.1, the action cannot be locally finite.

The next example shows that the action being locally finite is not a necessary condition for the semidirect product to be a Howson inverse semigroup.

**Example 5.2.** Let  $G$  be  $(\mathbb{Z}, +)$  and let  $E$  be  $\mathbb{Z}$  with the usual ordering. We consider the action  $\theta: G \rightarrow \text{Aut}(E)$  defined by  $\theta_n(m) = n + m$ . Then  $\theta$  is not locally finite but  $E *_\theta G$  is a Howson inverse semigroup.

It is straightforward to check that  $\theta$  is well defined and is not locally finite. Note also that  $G$ , being a free group, is a Howson group. (In fact,  $G$  is Howson as every nontrivial subgroup of  $\mathbb{Z}$  is actually cyclic.)

We say that an inverse subsemigroup  $S \leq E *_\theta G$  is *bounded* if there exists some  $M \in \mathbb{Z}$  such that  $m \leq M$  for every  $(m, n) \in S$ . It is easy to see that:

- if  $A = \{(m_1, n_1), \dots, (m_k, n_k)\}$  is a finite subset of  $E *_\theta G$  closed under inversion, then  $\langle A \rangle$  is bounded (with  $M = \max\{m_1, \dots, m_k\}$ );
- the intersection of bounded inverse subsemigroups of  $E *_\theta G$  is bounded.

Now we show that

$$\begin{aligned} \text{if } S \leq E *_\theta G \text{ is bounded and contains a nonidempotent,} \\ \text{then } S \text{ is finitely generated.} \end{aligned} \tag{7}$$

Given that  $S$  is by assumption inverse and contains a nonidempotent, the positive integer

$$N = \min\{n > 0: (m, n) \in S \text{ for some } m \in \mathbb{Z}\}$$

is well-defined. It follows easily from the division algorithm that

$$S \subseteq \mathbb{Z} \times N\mathbb{Z}$$

(notice that, if  $n, p > 0$ , then  $(k, n)^p = (k, np)$ ). For  $i = 0, \dots, N - 1$ , let

$$S_i = \{(m, n) \in S: m \equiv i \pmod{N}\}.$$

Then each nonempty  $S_i$  is an inverse subsemigroup of  $S$ . Therefore it suffices to show that each nonempty  $S_i$  is finitely generated.

Fixing such an  $i$ , and since  $S$  is bounded, we can define  $M_i = \max\{m : (m, n) \in S_i, n > 0\}$ . Let

$$S'_i = \{(m, 0) \in S_i : m > M_i\}.$$

We claim that

$$S_i = \langle \{(M_i, N)\} \cup S'_i \rangle. \quad (8)$$

We show first that  $(M_i, N) \in S_i$ . By definition of  $M_i$ , we have  $(M_i, n) \in S_i$  for some  $n > 0$ . On the other hand, we have  $(m, N) \in S$  for some  $m \in \mathbb{Z}$ . Since  $n > 0$ , there exists some  $k > 0$  such that  $nk + m \geq M_i$ . It follows easily that

$$(M_i, N) = (M_i, nk)(m, N)(M_i - nk, -nk) = (M_i, n)^k(m, N)(M_i, n)^{-k} \in S.$$

Now  $(M_i, n) \in S_i$  yields  $(M_i, N) \in S_i$ . Therefore  $\{(M_i, N)\} \cup S'_i \subseteq S_i$  and so

$$\langle \{(M_i, N)\} \cup S'_i \rangle \subseteq S_i.$$

Conversely, let  $(r, s) \in S_i$ . We may assume that  $s \geq 0$ . If  $r > M_i$ , then  $s = 0$  by maximality of  $M_i$  and so  $(r, s) \in S'_i$ . Thus we may assume that  $r \leq M_i$ . Since  $(r, s), (M_i, N) \in S_i$ , we may write  $(r, s) = (M_i - pN, qN)$  for some  $p, q \geq 0$ . Assuming that  $(M_i, N)^0$  denotes an absent factor, it follows that

$$\begin{aligned} (r, s) &= (M_i - pN, -pN)(M_i, (p+q)N) \\ &= (M_i - pN, -pN)(M_i, (p+q+1)N)(M_i - N, -N) \\ &= (M_i, N)^{-p}(M_i, N)^{p+q+1}(M_i, N)^{-1} \end{aligned}$$

and so  $S_i \subseteq \langle \{(M_i, N)\} \cup S'_i \rangle$ . Thus (8) holds. Since  $S$  is bounded,  $S'_i$  is finite and so each nonempty  $S_i$  is finitely generated. Therefore (7) holds.

Finally, we show that  $E *_\theta G$  is a Howson inverse semigroup. Let  $S, S'$  be finitely generated inverse subsemigroups of  $E *_\theta G$ . We may assume that  $S \cap S'$  is infinite, hence  $S$  and  $S'$  are both infinite. Since finitely generated semilattices are finite, then both  $S$  and  $S'$  contain nonidempotents, say  $(m, n)$  and  $(m', n')$ , respectively. Without loss of generality, we may assume that  $n, n' > 0$ . Hence  $(m, nn') = (m, n)^{n'} \in S$  and  $(m', nn') = (m', n')^n \in S'$ .

Suppose that  $S \cap S'$  contains only idempotents. By our previous remarks on boundedness, both  $S$  and  $S'$  are bounded and so is  $S \cap S'$ . Since  $S \cap S'$  is infinite, it follows that  $(r, 0) \in S \cap S'$  for some  $r \leq m, m'$ . Hence

$$(r, nn') = (r, 0)(m, nn') \in S, \quad (r, nn') = (r, 0)(m', nn') \in S',$$

and so  $S \cap S'$  would contain a nonidempotent, a contradiction. Therefore  $S \cap S'$  must contain a nonidempotent. Since  $S \cap S'$  is bounded, it follows from (7) that  $S \cap S'$  is finitely generated. Therefore  $E *_\theta G$  is a Howson inverse semigroup.

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